Normal and seminormal forms of sl_3-valued zero curvature representations

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Abstract

We find normal and seminormal forms for a sl_3-valued zero curvature representation (ZCR). We prove a theorem about reducibility of ZCR's, which says that if one of the matrix in a ZCR (A, B) falls to a proper subalgebra of sl_3, then the second matrix either falls to the same subalgebra or the ZCR is almost trivial. In the end of this paper we show examples of ZCR's and their normal forms.

1 Introduction

Zero curvature representations (ZCR) rank among the most important attributes of integrable partial differential equations [8]. A ZCR is usually treated as a special case of the Wahlquist–Estabrook prolongation structure [10], but the famous Wahlquist–Estabrook procedure is not sufficient for obtaining a complete classification of integrable systems. The main obstacle consists in the presence of a large group of gauge transformations. Thus we are naturally led to the problem of introduction of normal forms of ZCR's such that every orbit of the gauge action contains the corresponding normal form.

In nineties, independently M. Marvan [3] and S. Yu. Sakovich [6] introduced a characteristic element of a ZCR, which is a matrix that transforms by conjugation during gauge transformations of the ZCR. It follows that one can reduce the gauge freedom by putting the characteristic element in the Jordan normal form. There is a remaining gauge freedom, which can be used for further reduction of one of the matrices constituting the ZCR. This is rather similar to classification of pairs of matrices under simultaneous conjugation, developed by Belitskiĭ [1].

In case of the Lie algebra sl_2 a solution of the problem can be found in [4]. This made possible the subsequent complete classification of second-order evolution equations possessing an sl_2-valued ZCR [5].

In this work we try to obtain such a classification in case of sl_3. The number of possible normal forms is 8, compared to 2 in case of sl_2. As examples, we

consider the Tzitzéica equation [9], whose ZCR is known since 1910, Sawada-Kotera equation [7] and the Kupershmidt equation [2].

2 Preliminaries

Let us consider a system of nonlinear differential equations

$$F^l(t, x, u^k, \dots, u^k_I, \dots) = 0, \tag{1}$$

in two independent variables t and x, a finite number of dependent variables u^k and their derivatives u^k_I, where I denotes a finite symmetric multiindex over t and x.

Let J^∞ be an infinite-dimensional jet space such that t, x, u^k, u^k_I are local jet coordinates on J^∞. We have two distinguished vector fields on J^∞

$$D_t = \frac{\partial}{\partial t} + \sum_{k,I} u^k_{It} \frac{\partial}{\partial u^k_I}, \qquad D_x = \frac{\partial}{\partial x} + \sum_{k,I} u^k_{Ix} \frac{\partial}{\partial u^k_I},$$

which are called *total derivatives.* Let g be a matrix Lie algebra. By a g-valued *zero curvature representation* (ZCR) for (1) we mean two g-valued functions A, B which satisfy

$$D_t A - D_x B + [A, B] = 0 \tag{2}$$

as a consequence of (1). Let G be the connected and simply connected matrix Lie group associated with g. Then for every G-valued function W we define the *gauge transformation* of ZCR (A, B) by the formulas

$$\begin{aligned} A^W &:= D_x W \cdot W^{-1} + W \cdot A \cdot W^{-1} \\ B^W &:= D_t W \cdot W^{-1} + W \cdot B \cdot W^{-1} \end{aligned}$$

As is well known, (A^W, B^W) is a ZCR again, and we say that it is *gauge equivalent* to (A, B).

A *characteristic element* R is a g-valued function defined in [3]. The following assertion holds:

Proposition 2.1 ([3]) *Gauge equivalent ZCR's have conjugate characteristic elements.*

If a ZCR (A, B) is gauge equivalent to another ZCR with coefficients in a proper subalgebra of g, then we say that the ZCR is *reducible.* Otherwise it is said to be *irreducible.* A ZCR gauge equivalent to zero is called *trivial.* A very important case is a ZCR with coefficients in a non-solvable Lie algebra. The simplest case of a non-solvable Lie algebra is the algebra sl_2. In [4] the following proposition was obtained:

Proposition 2.2 *Let (A, B) be an irreducible sl_2-valued ZCR, let $R \neq 0$ be its characteristic element. Then we have one of the two following normal forms for*

R and A :
– *Nilpotent case*

$$R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}.$$

– *Diagonal case*

$$R = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}, \qquad A = \begin{pmatrix} a_1 & 1 \\ a_3 & -a_1 \end{pmatrix}.$$

3 Basic notions

In this section we explain the method to find the normal form of g-valued ZCR. The main idea is taken from the proposition 2.1. Gauge equivalent ZCR's have conjugate characteristic elements, therefore we can restrict ourselves to the characteristic elements in the Jordan normal form. Since the gauge transformation is a group action, it is possible to consider the stabilizer group of the characteristic element, which is a proper subgroup of G. The stabilizer is usually rather small (see Table 1), therefore we can compute its action on the matrix A and find the corresponding normal forms rather easily. We aim at finding the minimal set of normal forms. We can achieve substantial reduction by taking into account permutations of the Jordan blocks and using suitable automorphism of sl_3.

$$J_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix}; \quad \lambda_1 \neq \lambda_2, \qquad W_1 = \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & (w_1 w_2)^{-1} \end{pmatrix},$$

$$J_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}; \quad \lambda \neq 0, \qquad W_2 = \begin{pmatrix} w_{11} & w_{12} & 0 \\ w_{21} & w_{22} & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix},$$

$$J_3 = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}; \quad \lambda \neq 0, \qquad W_3 = \begin{pmatrix} w_1 & 0 & 0 \\ w_2 & w_1 & 0 \\ 0 & 0 & w_1^{-2} \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_4 = \begin{pmatrix} w_1 & 0 & 0 \\ w_2 & w_1 & w_3 \\ w_4 & 0 & w_1^{-2} \end{pmatrix},$$

$$J_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad W_5 = \begin{pmatrix} 1 & 0 & 0 \\ w_2 & 1 & 0 \\ w_3 & w_2 & 1 \end{pmatrix},$$

where $\quad Z = w_{11}w_{22} - w_{12}w_{21}.$

Table 1: Jordan forms and the corresponding stabilizers

In this work we distinguish between *normal forms* and *seminormal forms*. We say that we have the normal form if we have just a finite number of possibilities of the choice of the corresponding gauge matrix. If our choice of the corresponding gauge matrix depends on at least one arbitrary function, we say that we have the seminormal form. In this case we may use the residual gauge freedom to transform the matrix B.

Table 1 lists all possible Jordan forms J_i of sl_3-matrices and the corresponding stabilizers W_i, where w_j denote arbitrary complex numbers such that all algebraic operations make sense. J_2 and J_4 are degenerate cases of J_1 and J_3, respectively, when the two eigenvalues coincide and the dimension of the stabilizer raises from two to four.

4 Reducibility theorem

For further reference, we list here several subalgebras of sl_3. Two subalgebras a, b are said to be *conjugate*, if there exist $S \in GL_3$ such that $a = SbS^{-1}$. Note that for constant matrices $S \in SL_3$ conjugation and gauge equivalence coincide. Another obvious automorphism of sl_3 is $A \mapsto -A^\top$, which we call *transposition*. We introduce six permutation matrices

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In fact the permutation matrices P_1, P_3, P_5 have the determinant equal to -1, but this makes no harm since $-P_1, -P_3, -P_5 \in SL_3$ and $A^{P_i} = A^{-P_i}$.

In what follows, we frequently use six 6-dimensional subalgebras consisting of traceless matrices of either of the forms:

$$L_1 = \begin{pmatrix} . & 0 & 0 \\ . & . & . \\ . & . & . \end{pmatrix}, \quad L_2 = \begin{pmatrix} . & . & . \\ 0 & . & 0 \\ . & . & . \end{pmatrix}, \quad L_3 = \begin{pmatrix} . & . & . \\ . & . & . \\ 0 & 0 & . \end{pmatrix},$$

$$L_4 = \begin{pmatrix} . & . & . \\ 0 & . & . \\ 0 & . & . \end{pmatrix}, \quad L_5 = \begin{pmatrix} . & 0 & . \\ . & . & . \\ . & 0 & . \end{pmatrix}, \quad L_6 = \begin{pmatrix} . & . & 0 \\ . & . & 0 \\ . & . & . \end{pmatrix}.$$

These six subalgebras are mutually isomorphic via transposition or conjugation.

Theorem 4.1 *If the matrix A in the sl_3-valued ZCR (A, B) belongs to one of subalgebras $L_i, i = 1, \ldots, 6$, then the ZCR is either reducible or is gauge equivalent to one with $A = 0, D_x B = 0$.*

Proof. Recall that the ZCR (A,B) is reducible if both A, B fall to the same proper subalgebra or is gauge equivalent to such. Let the matrix A belongs to the subalgebra $L_1 = \{a_{12} = 0, a_{13} = 0\}$.

If $b_{13} = 0$ then from (2) we have $a_{23}b_{12} = 0$. If $b_{12} = 0$ then we are done. If $b_{12} \neq 0$ then $a_{23} = 0$. Now, if $b_{23} = 0$, then A falls to the same subalgebra L_6 as B and we are done. If $b_{23} \neq 0$ then we express from (2) stepwise all remaining elements of the matrix A:

$$\begin{aligned}
a_{11} &= (D_x b_{12} + (b_{12} D_x b_{23} - b_{23} D_x b_{12})/3b_{23})/b_{12},\\
a_{22} &= (b_{12} D_x b_{23} - b_{23} D_x b_{12})/(3b_{23}b_{12}),\\
a_{21} &= (b_{23}b_{12}^2 D_{tx} b_{23} + 2b_{23}^2 b_{12} D_{tx} b_{12} - b_{12}^2 D_x b_{23} D_t b_{23} - 2b_{23}^2 D_x b_{12} D_t b_{12}\\
&\quad -3b_{23}^2 b_{12}^2 D_x b_{11})/(3b_{12}^3 b_{23}^2),\\
a_{32} &= (2b_{23}b_{12}^2 D_{tx} b_{23} + b_{23}^2 b_{12} D_{tx} b_{12} - 2b_{12}^2 D_x b_{23} D_t b_{23} - b_{23}^2 D_x b_{12} D_t b_{12}\\
&\quad -3b_{23}^2 b_{12}^2 D_x b_{11} - 3b_{23}^2 b_{12}^2 D_x b_{22})/(3b_{23}^3 b_{12}^2),\\
a_{31} &= (b_{23}^2 b_{12}^3 D_{ttx} b_{23} + 2b_{23}^3 b_{12}^2 D_{ttx} b_{12} - 2b_{23} b_{12}^3 D_t b_{23} D_{tx} b_{23}\\
&\quad +b_{23}^2 b_{11} b_{12}^3 D_{tx} b_{23} - b_{23}^2 b_{12}^2 D_t b_{12} D_{tx} b_{23} - b_{23}^2 b_{12}^3 b_{22} D_{tx} b_{23}\\
&\quad -b_{23} b_{12}^3 D_x b_{23} D_{tt} b_{23} - 3b_{23}^3 b_{12}^3 D_{tx} b_{11} - 2b_{23}^3 b_{12}^2 b_{22} D_{tx} b_{12}\\
&\quad -6b_{23}^3 b_{12} D_t b_{12} D_{tx} b_{12} + 2b_{23}^3 b_{11} b_{12}^2 D_{tx} b_{12} - 2b_{23}^3 b_{12} D_x b_{12} D_{tt} b_{12}\\
&\quad +2b_{12}^3 D_x b_{23} (D_t b_{23})^2 + b_{23} b_{12}^2 D_x b_{23} D_t b_{23} D_t b_{12} + 6b_{23}^3 D_x b_{12} (D_t b_{12})^2\\
&\quad -b_{23} b_{11} b_{12}^3 D_x b_{23} D_t b_{23} + b_{23} b_{12}^3 b_{22} D_x b_{23} D_t b_{23} + 3b_{23}^3 b_{12}^2 D_x b_{11} D_t b_{12}\\
&\quad -2b_{23}^3 b_{11} b_{12} D_x b_{12} D_t b_{12} + 2b_{23}^3 b_{12} b_{22} D_x b_{12} D_t b_{12} - 3b_{23}^3 b_{12}^4 D_x b_{21}\\
&\quad -3b_{23}^3 b_{11} b_{12}^3 D_x b_{11} + 3b_{23}^3 b_{12}^3 b_{22} D_x b_{11} - 3b_{21} b_{23}^3 b_{12}^3 D_x b_{12})/(3b_{23}^4 b_{12}^4).
\end{aligned}$$

The gauge matrix which sends A to zero is then

$$W = \begin{pmatrix} b_{23}^{-1/3} b_{12}^{-2/3} & 0 & 0 \\ w_{21} & b_{12}^{1/3} b_{23}^{-1/3} & 0 \\ w_{31} & w_{32} & b_{12}^{1/3} b_{23}^{2/3} \end{pmatrix},$$

where

$$\begin{aligned}
w_{21} &= (-\tfrac{1}{3} D_t b_{23}/b_{23} - \tfrac{2}{3} D_t b_{12}/b_{12} + b_{11})/(b_{12}^{\frac{2}{3}} b_{23}^{\frac{1}{3}}),\\
w_{31} &= (-\tfrac{1}{3} D_{tt} b_{23}/b_{23} - \tfrac{2}{3} D_{tt} b_{12}/b_{12} + (\tfrac{2}{3} D_t b_{23}/b_{23})^2 + (\tfrac{10}{9} D_t b_{12}/b_{12})^2\\
&\quad +\tfrac{4}{9} D_t b_{23} D_t b_{12}/(b_{23}b_{12}) - \tfrac{2}{3} b_{11} D_t b_{23}/b_{23} + D_t b_{11}\\
&\quad -\tfrac{4}{3} b_{11} D_t b_{12}/b_{12} + b_{21} b_{12} + b_{11}^2)/(b_{12}^{\frac{2}{3}} b_{23}^{\frac{1}{3}}),\\
w_{32} &= (-\tfrac{2}{3} D_t b_{23}/b_{23} - \tfrac{1}{3} D_t b_{12}/b_{12} + b_{11} + b_{22}) b_{12}^{\frac{1}{3}}/b_{23}^{\frac{1}{3}}.
\end{aligned}$$

Then $D_x B = 0$ by equation (2).

Finally, if $b_{13} \neq 0$, then we apply the gauge transformation to the ZCR (A,B) with gauge matrix

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_{12}/b_{13} & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which keeps $a_{12} = 0, a_{13} = 0$, while b_{13} will become zero. $\square$

5 Normal and seminormal forms

We list here normal forms of sl_3-valued ZCR's with characteristic element in Jordan normal form in either of the forms $J_1, \ldots, J_5$. Matrices N_i^j denote normal forms, where the lower indices corresponds with Jordan normal forms. Dots denote arbitrary elements.

$$\textit{Case } J_1 \qquad N_1^1 = \begin{pmatrix} . & . & . \\ 1 & . & . \\ . & 1 & . \end{pmatrix}$$

$$\textit{Case } J_2 \qquad N_2^1 = \begin{pmatrix} 0 & 1 & 0 \\ . & . & 1 \\ . & . & . \end{pmatrix}$$

$$\textit{Case } J_3 \qquad N_3^1 = \begin{pmatrix} . & . & 1 \\ . & . & 0 \\ . & . & . \end{pmatrix}, \quad N_3^2 = \begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}$$

$$\textit{Case } J_4 \qquad N_4^1 = \begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}, \quad N_4^3 = \begin{pmatrix} 0 & 0 & 1 \\ . & 0 & 0 \\ . & . & 0 \end{pmatrix}$$

$$\textit{Case } J_5 \qquad N_5^1 = \begin{pmatrix} 0 & 0 & . \\ . & . & . \\ . & . & . \end{pmatrix}, \quad N_5^2 = \begin{pmatrix} 0 & . & 0 \\ . & . & . \\ . & 0 & . \end{pmatrix}$$

Theorem 5.1 *In a sl_3-valued ZCR such that its characteristic element has either of the Jordan normal forms $J_1, \ldots, J_5$, the matrix A has one of the above normal forms $N_1^1, N_2^1, N_3^1, N_3^2, N_4^1, N_4^3, N_5^1, N_5^2$, or A satisfies assumptions of Theorem 4.1.*

The remaining part of this paper is devoted to the proof of Theorem 5.1. In fact, we give an algorithm which assigns a normal or seminormal form to the matrix A. This algorithm proves Theorem 5.1. The symbol W_i^j denotes the corresponding gauge matrix which sends the matrix A to the normal (resp. seminormal) form N_i^j (resp. S_i^j).

5.1 Case J_1

The diagonal Jordan normal form is unique up to the order of the elements on the diagonal, i.e., up to conjugation with respect to one of the permutation

matrices $P_0, \ldots, P_5$. Given a matrix A, the corresponding gauge equivalent matrices will be $A_i = D_x P_i . P_i^{-1} + P_i A P_i^{-1} = P_i A P_i^{-1}$, $i = 0, 1, \ldots, 5$, namely

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{11} & a_{13} & a_{12} \\ a_{31} & a_{33} & a_{32} \\ a_{21} & a_{23} & a_{22} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ a_{12} & a_{13} & a_{11} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{pmatrix}, \quad A_4 = \begin{pmatrix} a_{33} & a_{31} & a_{32} \\ a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \end{pmatrix}, \quad A_5 = \begin{pmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{pmatrix}.$$

where $a_{33} = -a_{11} - a_{22}$ (since A is an sl_3 matrix).

Case 1. If there exists $i = 0, 1, \ldots, 5$ such that $a_{21} \neq 0$ and $a_{32} \neq 0$ in $A = A_i$, then we have

$$N_1^1 = \begin{pmatrix} . & . & . \\ 1 & . & . \\ . & 1 & . \end{pmatrix}, \quad W_1^1 = \begin{pmatrix} a_{32}^{\frac{1}{3}} a_{21}^{\frac{2}{3}} & 0 & 0 \\ 0 & a_{32}^{\frac{1}{3}} a_{21}^{\frac{-1}{3}} & 0 \\ 0 & 0 & a_{32}^{\frac{2}{3}} a_{21}^{\frac{-1}{3}} \end{pmatrix}.$$

One easily sees that the matrix W_1^1 is unique up to the choice of cubic roots, hence N_1^1 is a *normal form.*

Case 2. Otherwise, if there exists $i = 0, 1, \ldots, 5$ such that $a_{21} \neq 0$, $a_{32} = 0$ and $a_{31} \neq 0$ in $A = A_i$, then

$$N_1^2 = \begin{pmatrix} . & 0 & 0 \\ 1 & . & 0 \\ 1 & 0 & . \end{pmatrix}, \quad W_1^2 = \begin{pmatrix} a_{31}^{\frac{1}{3}} a_{21}^{\frac{1}{3}} & 0 & 0 \\ 0 & a_{31}^{\frac{1}{3}} a_{21}^{\frac{-2}{3}} & 0 \\ 0 & 0 & a_{21}^{\frac{1}{3}} a_{31}^{\frac{-2}{3}} \end{pmatrix}.$$

We have used the fact that $a_{12} = 0, a_{13} = 0$ and $a_{23} = 0$ as well. Indeed, if $a_{12} \neq 0$ (resp. $a_{13} \neq 0$, resp. $a_{23} \neq 0$) in A, then, using the permutation matrix P_3 (resp. P_4, resp. P_1), we would obtain the first case. N_1^2 is a normal form and belongs to the intersection of subalgebras L_5 and L_6.

The case when $a_{21} \neq 0$, $a_{32} = 0$, $a_{31} = 0$ and $a_{23} \neq 0$ in some $A = A_i$ can be converted to Case 2 by using conjugation by P_3 after transposition $A \mapsto -A^\top$.

Case 3. Otherwise, if there exists $i = 0, 1, \ldots, 5$ such that $a_{21} \neq 0$, $a_{32} = 0$, $a_{31} = 0$ and $a_{23} = 0$ in $A = A_i$, then

$$S_1^3 = \begin{pmatrix} . & . & 0 \\ 1 & . & 0 \\ 0 & 0 & . \end{pmatrix}. \quad W_1^3 = \begin{pmatrix} a_{21} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{21}^{-1} \end{pmatrix}.$$

Indeed, using the same argument as in Case 2 we may assume that $a_{13} = 0$. However, the most general gauge matrix is

$$\begin{pmatrix} a_{21} w_2 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & a_{21}^{-1} w_2^{-2} \end{pmatrix}$$

and depends on the choice of one arbitrary function w_2. If we set $w_2 = 1$, then we obtain W_1^3. Hence S_1^3 is a *seminormal form.* The matrix S_1^3 belongs to the intersection of subalgebras L_3 and L_6.

Case 4. If $a_{21} = 0$ for all A_i, then all the off-diagonal elements must be zero, therefore the seminormal form is

$$A = S_1^4 = \begin{pmatrix} . & 0 & 0 \\ 0 & . & 0 \\ 0 & 0 & . \end{pmatrix}.$$

The matrix S_1^4 belongs to the intersection of subalgebras L_1, L_2 and L_3.

5.2 Case J_2

The following case J_2 of Jordan normal form of characteristic element is singular and the number of parameters in the corresponding stabilizer subgroup increases from two to four (see Table 1). In this case we apply the automorphism $A \mapsto -A^\top$ to reduce the number of normal and seminormal forms.

Let $K = a_{13}D_x a_{23} - a_{23}D_x a_{13} + a_{11}a_{13}a_{23} - a_{21}a_{13}^2 + a_{12}a_{23}^2 - a_{22}a_{13}a_{23}$, $L = a_{32}D_x a_{31} - a_{31}D_x a_{32} + a_{11}a_{32}a_{31} + a_{21}a_{32}^2 - a_{12}a_{31}^2 - a_{22}a_{32}a_{31}$, and $R = a_{13}a_{31} + a_{23}a_{32}$.

Case 1. If $K \neq 0$, then the normal form is

$$N_2^1 = \begin{pmatrix} 0 & 1 & 0 \\ . & . & 1 \\ . & . & . \end{pmatrix}.$$

The corresponding gauge matrix W_2^1 is found to be

$$w_{11} = -a_{23}K^{-2/3}, \quad w_{12} = a_{13}K^{-2/3},$$
$$w_{21} = (\tfrac{2}{3}a_{23}K^{-1}D_x K - D_x a_{23} - a_{11}a_{23} + a_{13}a_{21})K^{-2/3},$$
$$w_{22} = (-\tfrac{2}{3}a_{13}K^{-1}D_x K + D_x a_{13} - a_{12}a_{23} + a_{22}a_{13})K^{-2/3}.$$

The case when $K = 0, L \neq 0$ can be reduced to Case 1. Indeed, using the automorphism $A \mapsto -A^\top$ we have $K \mapsto -L$ and $L \mapsto -K$.

Case 2. If $K = 0, L = 0, R \neq 0$, then

$$S_2^2 = \begin{pmatrix} . & 0 & 0 \\ 0 & . & . \\ 0 & 1 & . \end{pmatrix}, \quad W_2^2 = \begin{pmatrix} a_{23} & -a_{13} & 0 \\ a_{31}R^{-1/2} & a_{32}R^{-1/2} & 0 \\ 0 & 0 & R^{-1/2} \end{pmatrix}.$$

Indeed, applying W_2^2 to general sl_3 matrix A we obtain

$$A^{W_2^2} = \begin{pmatrix} . & -KR^{-1/2} & 0 \\ LR^{-3/2} & . & . \\ 0 & 1 & . \end{pmatrix},$$

and we see that for $K=0, L=0$ we have $A^{W_2^2} = S_2^2$. The seminormal form S_2^2 falls to the intersection of subalgebras L_1 and L_4.

For $K=0, L=0, R=0$ we have two subcases:

Case 3. If $a_{13} \neq 0$ or $a_{23} \neq 0$, then

$$S_2^3 = \begin{pmatrix} . & 0 & 0 \\ . & . & . \\ . & 0 & . \end{pmatrix}, \quad W_2^3 = \begin{pmatrix} a_{23} & -a_{13} & 0 \\ w_{21} & w_{22} & 0 \\ 0 & 0 & (w_{21}a_{13} + w_{22}a_{23})^{-1} \end{pmatrix}$$

for arbitrary nonzero parameters w_{21} and w_{22} such that $w_{21}a_{13} + w_{22}a_{23} \neq 0$. Indeed, applying W_2^3 to general sl_3 matrix A we obtain

$$A^{W_2^3} = \begin{pmatrix} . & K(w_{21}a_{13} + w_{22}a_{23})^{-1} & 0 \\ . & . & . \\ . & R(w_{21}a_{13} + w_{22}a_{23})^{-2} & . \end{pmatrix},$$

and we see that for $K=0, R=0$ we have $A^{W_2^3} = S_2^3$. Note that in this case $L = -K(a_{32}/a_{13})^2$ or $L = -K(a_{31}/a_{23})^2$. The seminormal form S_2^3 falls to the intersection of subalgebras L_1 and L_5.

When $a_{31} \neq 0$ or $a_{32} \neq 0$, then using the transposition $A \mapsto -A^\top$ we obtain Case 3.

Case 4. If $a_{13} = 0, a_{23} = 0, a_{31} = 0, a_{32} = 0$, then the seminormal form is

$$A = S_2^4 = \begin{pmatrix} . & . & 0 \\ . & . & 0 \\ 0 & 0 & . \end{pmatrix},$$

The seminormal form S_2^4 falls to the intersection of subalgebras L_3 and L_6.

5.3 Case J_3

In this case we use a modification of the permutation matrix P_3:

$$\overline{P_3} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Case 1. If $a_{13} \neq 0$, then

$$N_3^1 = \begin{pmatrix} . & . & 1 \\ . & . & 0 \\ . & . & . \end{pmatrix}, \quad W_3^1 = \begin{pmatrix} a_{13}^{-1/3} & 0 & 0 \\ -a_{23}a_{13}^{-4/3} & a_{13}^{-1/3} & 0 \\ 0 & 0 & a_{13}^{2/3} \end{pmatrix}.$$

N_3^1 is a normal form.

The case when $a_{13} = 0, a_{32} \neq 0$ can be reduced to Case 1 by using the conjugation by $\overline{P_3}$ after transposition $A \mapsto -A^\top$.

Case 2. If $a_{13} = 0, a_{32} = 0, a_{23} \neq 0, a_{12} \neq 0$, then

$$N_3^2 = \begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}, \quad W_3^2 = \begin{pmatrix} a_{23}^{-1/3} & 0 & 0 \\ (a_{23}a_{11} - \frac{1}{3}D_x a_{23})/(a_{12}a_{23}^{4/3}) & a_{23}^{-1/3} & 0 \\ 0 & 0 & a_{23}^{2/3} \end{pmatrix}.$$

N_3^2 is a normal form.

Case 3. If $a_{13} = 0, a_{32} = 0, a_{23} \neq 0, a_{12} = 0$, then

$$S_3^3 = \begin{pmatrix} . & 0 & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}, \quad W_3^3 = \begin{pmatrix} a_{23}^{-1/3} & 0 & 0 \\ 0 & a_{23}^{-1/3} & 0 \\ 0 & 0 & a_{23}^{2/3} \end{pmatrix}.$$

S_3^3 is a seminormal form and belongs to the intersection of subalgebras L_1 and L_5.

Case 4. If $a_{13} = 0, a_{32} = 0, a_{23} = 0, a_{31} \neq 0, a_{12} \neq 0$, then

$$N_3^4 = \begin{pmatrix} 0 & . & 0 \\ . & . & 0 \\ 1 & 0 & . \end{pmatrix}, \quad W_3^4 = \begin{pmatrix} a_{31}^{1/3} & 0 & 0 \\ (\frac{1}{3}D_x a_{31} + a_{31}a_{11})/(a_{12}a_{31}^{2/3}) & a_{31}^{1/3} & 0 \\ 0 & 0 & a_{31}^{-2/3} \end{pmatrix}.$$

N_3^4 is a normal form and belongs to the subalgebra L_6.

The case when $a_{13} = 0, a_{32} = 0, a_{23} = 0, a_{31} \neq 0, a_{12} = 0$ can be reduced to Case 3 by using the conjugation by $\overline{P_3}$ after transposition $A \mapsto -A^\top$ again.

Case 5. Otherwise, if $a_{13} = 0, a_{32} = 0, a_{23} = 0, a_{31} = 0, a_{12} \neq 0$, then

$$S_3^5 = \begin{pmatrix} 0 & . & 0 \\ . & . & 0 \\ 0 & 0 & . \end{pmatrix}, \quad W_3^5 = \begin{pmatrix} 1 & 0 & 0 \\ a_{11}a_{12}^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

S_3^5 is a seminormal form and belongs to the intersection of subalgebras L_3 and L_6.

Case 6. Otherwise, if $a_{13} = 0, a_{32} = 0, a_{23} = 0, a_{31} = 0, a_{12} = 0$, then the seminormal form is

$$A = S_3^6 = \begin{pmatrix} . & 0 & 0 \\ . & . & 0 \\ 0 & 0 & . \end{pmatrix}.$$

Matrices of this form constitute a 3-dimensional solvable subalgebra of sl_3.

5.4 Case J_4

The Case J_4 is singular again (see Case J_2). We use again the modified permutation matrix $\overline{P_3}$.

Let $M = a_{12}D_x a_{13} - a_{13}D_x a_{12} - 2a_{12}a_{13}a_{22} + a_{23}a_{12}^2 - a_{32}a_{13}^2 - a_{11}a_{12}a_{13}$ and $N = a_{12}D_x a_{32} - a_{32}D_x a_{12} + 2a_{11}a_{12}a_{32} - a_{31}a_{12}^2 + a_{13}a_{32}^2 + a_{12}a_{22}a_{32}$.

Case 1. If $a_{12} \neq 0, M \neq 0$, then the normal form is

$$N_4^1 = \begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}.$$

The corresponding gauge matrix W_4^1 is obtained in the following way:

$$w_1 = a_{12}^{2/3}/M^{1/3}, \qquad w_2 = (D_x w_1 + a_{11} w_1)/a_{12},$$

$$w_3 = w_1 a_{13}/a_{12}, \qquad w_4 = -a_{32}/(w_1^2 a_{12}).$$

The case when $a_{12} \neq 0, M = 0, N \neq 0$ may be reduced to Case 1 by using the conjugation by $\overline{P_3}$ after transposition $A \mapsto -A^\top$.

Case 2. If $a_{12} \neq 0, M = 0, N = 0$, then

$$S_4^2 = \begin{pmatrix} 0 & . & 0 \\ . & . & 0 \\ 0 & 0 & . \end{pmatrix}, \quad W_4^2 = \begin{pmatrix} 1 & 0 & 0 \\ a_{11}a_{12}^{-1} & 1 & a_{13}a_{12}^{-1} \\ -a_{32}a_{12}^{-1} & 0 & 1 \end{pmatrix}.$$

Indeed, applying W_4^2 to general sl_3 matrix A we obtain

$$A^{W_4^2} = \begin{pmatrix} 0 & . & 0 \\ . & . & M/a_{12}^2 \\ -N/a_{12}^2 & 0 & . \end{pmatrix},$$

and we see that for $M = 0, N = 0$ we have $A^{W_4^2} = S_4^2$. The seminormal form S_4^2 falls to the intersection of subalgebras L_3 and L_6.

Case 3. If $a_{12} = 0, a_{13} \neq 0, a_{32} \neq 0$, then the normal form is

$$N_4^3 = \begin{pmatrix} 0 & 0 & 1 \\ . & 0 & 0 \\ . & . & 0 \end{pmatrix}.$$

The corresponding gauge matrix W_4^3 is obtained in the following way:

$$w_1 = a_{13}^{-1/3}, \qquad w_3 = (D_x a_{13} - 3a_{13}a_{22})/3a_{32}a_{13}^{4/3},$$

$$w_4 = -(D_x a_{13} - 3a_{11}a_{13})/3a_{13}^{4/3},$$

$$w_2 = -\frac{w_3 D_x a_{13}}{3a_{13}^2} - \frac{D_x w_3 - a_{32}a_{13}^{1/3}w_3^2 - a_{11}w_3 - 2a_{22}w_3 + a_{23}a_{13}^{-1/3}}{a_{13}}.$$

Case 4. If $a_{12} = 0, a_{13} = 0, a_{32} \neq 0$, then

$$S_4^4 = \begin{pmatrix} . & 0 & 0 \\ . & . & . \\ 0 & 1 & 0 \end{pmatrix}, \quad W_4^4 = \begin{pmatrix} a_{32}^{1/3} & 0 & 0 \\ \dfrac{a_{31}}{a_{32}^{2/3}} & a_{32}^{1/3} & -\dfrac{2D_x a_{32} + 3a_{32}a_{11} + 3a_{32}a_{22}}{3a_{32}^{5/3}} \\ 0 & 0 & a_{32}^{-2/3} \end{pmatrix}.$$

The seminormal form S_4^4 falls to the subalgebra L_1.

The case when $a_{12} = 0, a_{13} \neq 0, a_{32} = 0$ is reducible to Case 4 by using the conjugation by $\overline{P_3}$ after transposition $A \mapsto -A^\top$.

Case 5. If $a_{12} = 0, a_{13} = 0, a_{32} = 0, a_{23} \neq 0$, then

$$S_4^5 = \begin{pmatrix} . & 0 & 0 \\ 0 & . & . \\ . & 0 & . \end{pmatrix}, \quad W_4^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{21}a_{23}^{-1} & 0 & 1 \end{pmatrix}.$$

The seminormal form S_4^5 falls to the intersection of subalgebras L_1 and L_5.

Case 6. If $a_{12} = 0, a_{13} = 0, a_{32} = 0, a_{23} = 0$, then the seminormal form is

$$A = S_4^6 = \begin{pmatrix} . & 0 & 0 \\ . & . & 0 \\ . & 0 & . \end{pmatrix}.$$

The seminormal form S_4^6 falls to the intersection of subalgebras L_5 and L_6.

5.5 Case J_5

In this case we use a modification of the permutation matrix P_5:

$$\overline{P_5} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Case 1. If $a_{13} \neq 0$, then

$$N_5^1 = \begin{pmatrix} 0 & 0 & . \\ . & . & . \\ . & . & . \end{pmatrix}, \quad W_5^1 = \begin{pmatrix} 1 & 0 & 0 \\ a_{12}a_{13}^{-1} & 1 & 0 \\ a_{11}a_{13}^{-1} & a_{12}a_{13}^{-1} & 1 \end{pmatrix}$$

N_5^1 is a normal form.

Case 2. Otherwise, if $a_{13} = 0, a_{12} \neq 0$, then

$$N_5^2 = \begin{pmatrix} 0 & . & 0 \\ . & . & . \\ . & 0 & . \end{pmatrix}, \quad W_5^2 = \begin{pmatrix} 1 & 0 & 0 \\ a_{11}a_{12}^{-1} & 1 & 0 \\ w_3 & a_{11}a_{12}^{-1} & 1 \end{pmatrix},$$

where $w_3 = (a_{11}D_x a_{12} - a_{12}D_x a_{11} + a_{23}a_{11}^2 - a_{32}a_{12}^2 - 2a_{22}a_{12}a_{11} - a_{12}a_{11}^2)/a_{12}^3$. N_5^2 is a normal form.

The case when $a_{13} = 0, a_{12} = 0, a_{23} \neq 0$ is reducible to Case 2 by using the conjugation by $\overline{P_5}$ after transposition $A \mapsto -A^\top$.

Case 3. Otherwise, if $a_{13} = 0, a_{12} = 0, a_{23} = 0$, then the seminormal form is

$$A = S_5^3 = \begin{pmatrix} . & 0 & 0 \\ . & . & 0 \\ . & . & . \end{pmatrix}.$$

Matrices of this form fall to the intersection of subalgebras L_1 and L_6.

6 Examples

Example 6.1 The Tzitzéica equation [9]:

$$u_{tx} = e^u - e^{-2u}.$$

The corresponding ZCR, which depends on a parameter $m \neq 0$, is

$$A = \begin{pmatrix} -u_x & 0 & m \\ m & u_x & 0 \\ 0 & m & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \mathrm{e}^{-2u}/m & 0 \\ 0 & 0 & \mathrm{e}^u/m \\ \mathrm{e}^u/m & 0 & 0 \end{pmatrix}.$$

The matrix A belongs to the Case J_1 with the normal form N_1^1. Namely, the Jordan normal form of the characteristic element R and the matrix $A^{W_1^1}$ are

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^{W_1^1} = \begin{pmatrix} -u_x & 0 & m^3 \\ 1 & u_x & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 6.2 The Sawada-Kotera equation [7]:

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x$$

The corresponding ZCR and the corresponding normal form belongs to the Case J_4. The matrix A has the normal form N_4^1, namely,

$$A = \begin{pmatrix} 0 & -1 & 0 \\ u & 0 & -m \\ 1 & 0 & 0 \end{pmatrix}, \quad A^{W_4^1} = \begin{pmatrix} 0 & -1 & 0 \\ u & 0 & 1 \\ -m & 0 & 0 \end{pmatrix}.$$

The matrix B is rather large, hence omitted.

Example 6.3 The Kupershmidt equation [2]:

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x$$

The corresponding ZCR and the corresponding normal form belongs to the Case J_5. The matrix A has the normal form N_5^2, namely,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -u & 0 & 1 \\ m & -u & 0 \end{pmatrix}, \quad A^{W_5^2} = \begin{pmatrix} 0 & 1 & 0 \\ -2u & 0 & 1 \\ u_x + m & 0 & 0 \end{pmatrix}.$$

The matrix B is large, hence omitted again.

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